International Journal of Theoretical Physics, Vol. 42, No. 1, January 2003 (© 2003)

Planck's Spectral Distribution Law in *N* Dimensions

Sami M. Al-Jaber[1]

Received May 17, 2002

Planck's spectral distribution law is derived in N-dimensional space. Some relevant formulas are thus obtained and analyzed. The behavior of these formulas in higher dimensions is examined and some numerical values are calculated.

KEY WORDS: black body radiation; boson systems.

1. INTRODUCTION

Today it is well recognized that the concept of dimensions plays an important role in many areas of physics. Fukutaka and Kashiwa (1987) and Neves and Wotzasek (2000) considered path integrals formulation and free particle quantization on N-dimensional sphere. Some authors discussed hypercubic lattices in N dimensions (Bonnier, 2001; Francesco and Guitter, 2002; Joyce and Zucker, 2001; Miller and Srivastava, 2001; Shrock and Wu, 2000). Recent work in mathematical physics has been reported: Phase space and momentum operators in N dimensions (Bashir *et al.*, 2001; Paz, 2001), topological aspect of topological defects in arbitrary dimensions (Jiang, 2000), superintegrability and exact solvability models in arbitrary dimensions (Rodriguez, 2002). A great deal of work has been carried out in quantum gravity theories in extra dimensions. Arkani-Hamed (2002) considered approximate symmetries from distant breaking in extra dimensions. Biesiada and Malec (2002) discussed the white dwarf cooling from extra dimensions. Oda (2001) and Bander (2001) studied the generalization of locally localized gravity models to higher dimensions. The investigation of Einstein gravitational equations of motion in higher dimensions was recently reported (Ito, 2001; Ivashchuk and Melnikov, 2000; Mannheim, 2001). Furthermore, Al-Jaber (1999) considered Fermi gas in D dimensions and Salasnich (2000) investigated ideal quantum gases in D-dimensional space and confined in power-law potentials.

[1] Department of Physics, An-Najah National University, Nablus, West Bank, Palestine, via Israel; e-mail: jaber@najah.edu.

0020-7748/03/0100-0111/0 © 2003 Plenum Publishing Corporation

The present paper further studies the photon gas in N-dimensional space. In particular, we derive the Planck's spectral distribution law and thus examine Wien's displacement constant and Stephan–Boltzmann law in N dimensions.

2. THE PHOTON GAS AND PLANCK'S LAW IN *N* DIMENSIONS

We consider electromagnetic radiation in thermal equilibrium within an enclosure of volume V in N-dimensional space, thus our system can be considered a photon gas. Each photon is a boson of zero mass and has $(N-1)$ states of polarization. The average number of photons in each energy level at temperature T is

$$\langle n_i \rangle = \left[\exp(\beta E_i) - 1 \right]^{-1}. \tag{1}$$

If the volume V is large then the energy levels are closely spaced and E_i can be considered as a continuous variable E. Thus number of photons $n(E, T)\,dE$ within the energy range E to $E + dE$ at temperature T is

$$n(E, T)dE = D(E)[\exp(\beta E) - 1]^{-1} dE, \tag{2}$$

where *D(E)* is the density of states (number of photon states per unit energy interval). We may express the density of states as function of wavelength, λ, as

$$G(\lambda) = D(E)|dE/d\lambda|, \tag{3}$$

and therefore the number of photons within the wavelength interval λ and $\lambda + d\lambda$ is given by

$$F(\lambda, T)\,d\lambda = G(\lambda)[\exp(2\pi\beta\hbar c/\lambda) - 1]^{-1} d\lambda. \tag{4}$$

Thus the energy density within the interval $d\lambda$ is

$$\rho(\lambda, T)\,d\lambda = EF(\lambda, T)\,d\lambda/V = 2\pi\hbar c F(\lambda, T)\,d\lambda/(\lambda V). \tag{5}$$

Our aim now is to find *D(E)* and then $G(\lambda)$. The wave function of each photon is a plane wave given, in N-dimensional space, by

$$\Psi(x_1, x_2, \ldots, x_N) = \prod_{j=1}^{N} \exp(ik_j x_j), \tag{6}$$

and if periodic boundary conditions are imposed, then

$$k_j = \frac{2\pi}{L} n_j, \quad j = 1, 2, \ldots, N, \tag{7}$$

where n_j are integers, and L is the side length of our cubic volume V. The number of states, up to a given value of k, is equal to number of unit cells of unit volume within a hypersphere of radius k. Taking into account that there are $(N-1)$ states of polarization of each photon, the total number of photon states, n, within the hypersphere is

$$n = (N-1)\left(\frac{L}{2\pi}\right)^N V_N, \tag{8}$$

where V_N is the volume of the hypersphere given by (Bender, 1995)

$$V_N = \frac{\pi^{N/2} K^N}{\Gamma(1+N/2)}, \tag{9}$$

therefore,

$$n = (N-1)(L/2\pi)^N \frac{\pi^{N/2} K^N}{\Gamma(1+N/2)}, \tag{10}$$

and upon using $V = L^N$ and $K = E/\hbar c$, we get

$$n = \frac{(N-1)V}{2^N \pi^{N/2} \Gamma(1+N/2)} (E/\hbar c)^N. \tag{11}$$

The density of states is

$$D(E) = \frac{dn}{dE} = \frac{(N-1)VE^{N-1}}{2^{N-1}\pi^{N/2}\Gamma(N/2)(\hbar c)^N}, \tag{12}$$

and thus, upon the use of Eq. (3), we have

$$G(\lambda) = \frac{2(N-1)V\pi^{N/2}}{\Gamma(N/2)\lambda^{N+1}}. \tag{13}$$

Therefore, Eq. (4) yields

$$F(\lambda, T)\, d\lambda = \frac{2(N-1)V\pi^{N/2} d\lambda}{\Gamma(N/2)\lambda^{N+1}} [\exp(hc/\lambda kT) - 1]^{-1}, \tag{14}$$

and hence the spectral distribution function, Eq. (5), is

$$\rho(\lambda, T) = \frac{2(N-1)\pi^{N/2} hc}{\Gamma(N/2)\lambda^{N+2}} \frac{1}{e^{hc/\lambda kT} - 1}. \tag{15}$$

This is the Planck's spectral distribution law in N dimensions, that is, the energy density per unit wavelength. The total energy density is

$$\rho(T) = \int_0^\infty \rho(\lambda, T)\, d\lambda = \frac{2(N-1)\pi^{N/2} hc}{\Gamma(N/2)} \int_0^\infty \frac{d\lambda}{\lambda^{N+2}(e^{hc/\lambda kT} - 1)}, \tag{16}$$

and if we let $x = hc/\lambda kT$, then we get

$$\rho(T) = \frac{2(N-1)\pi^{N/2}(kT)^{N+1}}{\Gamma(N/2)(hc)^N} \int_0^\infty \frac{x^N dx}{e^x - 1}. \tag{17}$$

The integral in Eq. (17) is (Dwight, 1961)

$$\int_0^\infty \frac{x^N dx}{e^x - 1} = N!\xi(N+1), \tag{18}$$

where $\xi(N+1)$ is the Riemann zeta function.

The substitution of Eq. (18) into Eq. (17) and using $N! = \Gamma(N+1)$, gives us

$$\rho(T) = \frac{2(N-1)\pi^{N/2}k^{N+1}\Gamma(N+1)}{\Gamma(N/2)(hc)^N}\xi(N+1)T^{N+1}. \tag{19}$$

It is clear the dependence of the total energy density on the dimension N. It is worthy to mention that in the three-dimensional case, our result yields (for $N = 3$) the expected result (Bransden and Joachain, 2000)

$$\rho(T) = \frac{8\pi^5 k^4}{15h^3c^3}T^4, \tag{20}$$

where we used $\xi(4) = \pi^4/90$ (Abramowitz and Stegun, 1972).

It is interesting to calculate the total number of black body photons per unit volume at absolute temperature

$$n = \int_0^\infty \frac{\rho(\lambda, T)\, d\lambda}{hc/\lambda}. \tag{21}$$

Using Eq. (15) and, as before, $x = hc/\lambda kT$ yields

$$n = \frac{2(N-1)(N-1)!\pi^{N/2}\xi(N)}{\Gamma(N/2)(hc)^N}K^N T^N, \tag{22}$$

which could be written as

$$n = d_N T^N, \tag{23}$$

with d_N being the coefficient of T^N in Eq. (22). One sees that the number of photons per unit volume increases as the dimension N increases. For numerical values, Table I shows the values of the parameter d_N for several values of the dimension N. It is observed that for the three-dimensional case ($N = 3$) our results gives d_3 that coincides with the well-known result (Bransden and Joachain, 2000).

It may also be tempting to calculate the average energy $\langle E \rangle$ of a black body photon at absolute temperature. This is readily deduced by dividing the total energy density $\rho(T)$, given by Eq. (19), by the total number of photons per unit volume, given by Eq. (22). The result, by doing so, is

$$\langle E \rangle = \frac{N\xi(N+1)}{\xi(N)}kT = a_N T, \tag{24}$$

where we have used $\Gamma(N+1) = N(N-1)!$. For numerical values, Table II shows the values of a_N for several values of the dimension N.

Table I. Values of the Parameter d_N for Different Values of N

N	2	3	4	5	6	7	8	9	10
d_N	2.81×10^{4}	2.02×10^{7}	8.94×10^{9}	4.23×10^{12}	2.12×10^{15}	1.12×10^{18}	6.22×10^{20}	6.38×10^{23}	2.17×10^{26}

Table II. Values of a_N for Different Values of N

N	2	3	4	5	6	7	8	9	10
$a_N(10)^{23}$	2.018	3.729	5.291	6.773	8.211	9.624	11.021	12.415	13.80

It is observed from Table II and Eq. (24) that, at a given temperature, the average energy of a photon increases as the dimension N increases. Again, our results give the expected result for the $N = 3$ case, i.e., a_3 (Bransden and Joachain, 2000).

3. SPECIAL CASES

In this section we discuss two special cases that are relevant to Planck's radiation law in N dimensions.

3.1. The Long Wavelength Limit

In the long wavelength limit, we can keep the first two terms in the expansion of the exponential in the denominator of Eq. (15) and we get

$$\lim_{\text{large } \lambda} \rho(\lambda, T) = \frac{2(N-1)\pi^{N/2}}{\Gamma(N/2)\lambda^{N+1}} kT, \tag{25}$$

which is Rayleigh–Jeans formula in N dimensions. For the $N = 3$ case, the result in Eq. (25) yields the expected formula in the three-dimensional case, namely

$$\lim_{\text{large } \lambda} \rho(\lambda, T) \overset{N=3}{=\!\!\longrightarrow} \frac{8\pi}{\lambda^4} kT. \tag{26}$$

3.2. Wein's Displacement Law

We need to find the value of λ for which the Planck spectral distribution, Eq. (15), is maximum. The requirement that $d\rho/dt(\lambda, T) = 0$ gives

$$\lambda T = \frac{hc}{(N+2)k}[1 - \exp(-hc/\lambda kT)]^{-1}. \tag{27}$$

By letting $x = hc/\lambda kT$, Eq. (27) becomes

$$x = (N+2)(1 - e^{-x}), \tag{28}$$

and upon writing $x = (N+2) - \varepsilon$, the above equation can be written as

$$(N+2) - \varepsilon = (N+2)\left(1 - e^{-(N+2)} e^{\varepsilon}\right). \tag{29}$$

By expanding $e^{\varepsilon} \approx 1 + \varepsilon + \varepsilon^2/2$, we find

$$\frac{\varepsilon^z}{2} + \varepsilon\left(1 - \frac{e^{N+2}}{N+2}\right) + 1 = 0, \tag{30}$$

whose positive root is

$$\varepsilon = \left(\frac{e^{N+2}}{N+2} - 1\right) - \left[\left(1 - \frac{e^{N+2}}{N+2}\right)^2 - 2\right]^{1/2}. \tag{31}$$

The defining equation $x = hc/\lambda kT$ implies that

$$\lambda_{\max} T = \frac{hc}{k(N+2-\varepsilon)} \equiv b_N, \tag{32}$$

which is Wien's displacement law in N dimensions and b_N is Wien's displacement constant. $\lambda_{\max}$ is the wavelength at which $\rho(\lambda, T)$ has its maximum value for a given absolute temperature. Using Eq. (15), we have

$$\rho(\lambda_{\max}, T) = \frac{2(N-1)hc\pi^{N/2}}{\Gamma(N/2)(b_N)^{N+2}} \frac{T^{N+2}}{e^x - 1}. \tag{33}$$

It is instructive to calculate ε, x, b_N, and $\rho(\lambda_{\max}, T)$ for several values of N. This is given in Table III.

It is noticed that as the dimension N increases b_N decreases, which implies that $\lambda_{\max}$ shifts toward lower values. This means that the system radiates at high energy. This result is consistent with what was found in Section 2 that the average energy of each photon increases as N increases. It is also observed from Table III that as N increases $\rho(\lambda_{\max}, T)$, at a given temperature, increases. This is also consistent with our result in Section 2, which shows that the number of photons per unit volume increases as N increases.

Table III. Values of ε, x, b_N, and $\rho(\lambda_{\max}, T)$ for Several Values of N

N	ε	x	$b_N(10^{-3})$	$\rho(\lambda_{\max}, T)$
2	0.07930	3.9207	3.6728	$1.389 \times 10^{-16}T^4$
3	0.03488	4.965	2.9002	$1.712 \times 10^{-13}T^5$
4	0.01509	5.9849	2.4061	$1.532 \times 10^{-10}T^6$
5	0.00642	6.9936	2.0590	$1.226 \times 10^{-7}T^7$
6	0.00269	7.9973	1.8006	$9.396 \times 10^{-5}T^8$
7	0.00111	8.9989	1.6002	$7.093 \times 10^{-2}T^9$
8	0.00045	9.9995	1.4401	$5.365 \times 10^{1}T^{10}$
9	0.00018	10.9998	1.3091	$4.082 \times 10^{4}T^{11}$
10	0.00007	11.9999	1.2000	$3.151 \times 10^{7}T^{12}$

4. DISCUSSION AND CONCLUSIONS

In summary, the Planck's spectral distribution law in N-dimensional space is derived and thus the energy density was obtained. Afterwards the total number of photons per unit volume was calculated and was shown to be proportional to the Nth power of absolute temperature. Numerical values of the proportionality factor d_N were calculated and was shown that this factor increases as the dimension N increases. Subsequently, the average energy of each photon was obtained and was shown that it increases as N increases. Furthermore, Rayleigh–Jeans formula in N dimensions is obtained when the large wavelength limit of the spectral distribution function is taken. Finally, Wien's displacement law was derived and numerical values of Wien's displacement constant in higher dimensions were given. It is shown that this constant decreases as the dimension N increases, which implies that the wavelength at which the spectral distribution function is maximum shifts toward lower values. This means that the photon gas radiates at high energy in higher dimensions, which is consistent with our result concerning the average energy of each photon. It is also noticed that the maximum values of the spectral distribution function, at a given temperature, increases as N increases, which is consistent with our result for the number of photons per unit volume in higher dimensions. We emphasize that our main results yield the expected results for the three-dimensional space.

REFERENCES

Abramowitz, M. and Stegun, I. E. (1972). *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, DC.
Al-Jaber, S. M. (1999). *International Journal of Theoretical Physics* **38**, 919.
Arkani-Hamed, N. (2002). *Physical Review D* **65**, 6003.
Bander, M. (2001). *Physical Review D* **64**, 5021.
Bashir, A., Delbourgo, R., and Roberts, M. L. (2001). *Journal of Mathematical Physics* **42**, 5553.
Bender, C. M. (1995). *American Journal of Physics* **63**, 1011.
Biesiada, M. and Malec, B. (2002). *Physical Review D* **65**, 3008.
Bonnier, B. (2001). *Journal of Physics A* **34**, 10757.
Bransden, B. H., and Joachain, C. J. (2000). *Quantum Mechanics*, Prentice-Hall, England.
Dwight, H. B. (1961). *Tables of Integrals and Other Mathematical Data*, Macmillan, New York.
Francesco, P. and Guitter, E. (2002). *Journal of Physics A* **35**, 897.
Fukutaka, H. and Kashiwa, T. (1987). *Annals of Physics* **176**, 301.
Ito, M. (2001). *Physical Review D* **64**, 4021.
Ivashchuk, V. D., and Melnikov, V. N. (2000). *Journal of Mathematical Physics* **41**, 6341.
Jiang, Y. (2000). *Journal of Mathematical Physics* **41**, 6463.
Joyce, G. S. and Zucker, I. J. (2001). *Journal of Physics A* **34**, 7349.
Mannheim, D. (2001). *Physical Review D* **64**, 8501.
Miller, R. A. and Srivastava, H. M. (2001). *Journal of Physics A* **34**, 2777.
Neves, C. and Wotzasek, C. (2000). *Journal of Physics A* **33**, 6447.

Oda, I. (2001). *Physical Review D* **64**, 6002.
Paz, G. (2001). *European Journal of Physics* **22**, 337.
Rodriguez, A. M. (2002). *Journal of Mathematical Physics* **43**, 1309.
Salasnich, L. (2000). *Journal of Mathematical Physics* **41**, 8016.
Shrock, R. and Wu, F. Y. (2000). *Journal of Physics A* **33**, 3881.